ON THE FERMIONIC FORMULA AND THE KIRILLOV-RESHETIKHIN CONJECTURE.

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0. Introduction

The irreducible finite-dimensional representations of quantum affine algebras $\mathbf{U}_q(\hat{\mathfrak{g}})$ have been studied from various viewpoints, [AK], [CP1], [CP3], [C], [CP4], [FR], [FM], [KR], [K]. These representations decompose as a direct sum of irreducible representations of the quantized eneveloping algebra $\mathbf{U}_q(\mathfrak{g})$ associated to the underlying finite-dimensional simple Lie algebra $\mathfrak{g}$. But, except in a few special cases, little is known about the isotypical components occuring in the decomposition. However, for a certain class of modules (namely the one associated in a canonical way to a multiple of a fundamental weight of $\mathfrak{g}$), there is a conjecture due to Kirillov and Reshetikhin [KR] for Yangians that describes the $\mathfrak{g}$-isotypical components. A combinatorial interpretation of their conjecture was given by Kleber, [Kl] (see also [HKOTY]). It is the purpose of this paper to prove the conjecture for the quantum affine algebras associated to the classical simple Lie algebras, using Kleber's interpretation.

We now describe the conjecture and the results more explicitly. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be a set of fundamental weights for $\mathfrak{g}$ and, for any dominant integral weight μ, let $V_q(\mu)$ denote the irreducible $\mathbf{U}_q(\mathfrak{g})$-module with highest weight μ. For each $m \in \mathbf{Z}^+$ and $i = 1, \cdots, n$, the conjecture predicts the existence of an irreducible representation $V_q^{\mathrm{aff}}(m\lambda_i)$ of $\mathbf{U}_q(\hat{\mathfrak{g}})$ whose highest weight when viewed as a representation of $\mathbf{U}_q(\mathfrak{g})$ is $m\lambda_i$. The decomposition of the tensor product of N such representations as $\mathbf{U}_q(\mathfrak{g})$–modules is given by,

$$\bigotimes_{a=1}^{N} V_q^{\mathrm{aff}}(m_a \lambda_{i_a}) \simeq \sum_{\lambda} n_{\lambda}, V_q(\lambda)$$

where, the sum runs over all dominant integral weights $\lambda \leq \sum m_a \lambda_{i_a}$. The nonnegative integer n_{λ} is the multiplicity with which the irreducible $\mathbf{U}_q(\mathfrak{g})$-module $V_q(\lambda)$ occurs in the decomposition. Write $\lambda = \sum m_a \lambda_{i_a} - \sum n_i \alpha_i$, $n_i \in \mathbf{Z}^+$. Then

$$n_{\lambda} = \sum_{\text{partitions}} \prod_{n \geq 1} \prod_{k=1}^{r} \binom{P_n^{(k)}(\nu) + \nu_n^{(k)}}{\nu_n^{(k)}}$$

The sum is taken over all ways of choosing partitions $\nu^{(1)}, \ldots, \nu^{(r)}$ such that $\nu^{(i)}$ is a partition of n_i which has $\nu_n^{(i)}$ parts of size n (so $n_i = \sum_{n \geq 1} n\nu_n^{(i)}$). The function

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P is defined by

$$P_n^{(k)}(\nu) \;=\; \sum_{a=1}^{N} \min(n, m_a)\delta_{k,l_a} - 2\sum_{h\geq 1}\min(n,h)\nu_h^{(k)} +$$

$$+\sum_{j\neq k}^{r}\sum_{h\geq 1}\min(-a_{k,j}n, -c = a_{j,k}h)\nu_h^{(j)}$$

where $A = (a_{i,j})$ is the Cartan matrix of $\mathfrak{g}$, and $\binom{a}{b} = 0$ whenever $a < b$.

The formula describing the n_λ is called the fermionic formula, the connection with representation theory was made by Kirillov and Reshetikhin. They outlined a proof (using the techniques of the Bethe ansatz) of the conjecture when $\mathfrak{g}$ is of type A_n, and showed that the module $V_q^{\mathrm{aff}}(m\lambda_i)$ must be isomorphic as an A_n–module to $V_q(m\lambda_i)$. A rigorous mathematical proof was given recently in [KSS].

For other simple Lie algebras, the conjecture remained open, one reason being that the fermionic formula is not very tractable computationally, even in very simple cases. Although candidates were known for the modules in the case $N = m = 1$, [CP3], it was impossible to verify the conjectures. Kirillov and Reshetikhin did conjecture (when $N = 1$) a more explicit description of the multiplicities given by the fermionic formula. For instance, when $\mathfrak{g}$ is an even orthogonal algebra, and λ_i does not correspond to the spin nodes, then they conjectured that the multiplicity of $V_q(\lambda)$ in $V_q^{\mathrm{aff}}(m\lambda_i)$ satisfies $n_\lambda \leq 1$ and

$$(0.1) \qquad n_\lambda \neq 0 \quad \text{iff} \quad \lambda = \sum_{j\geq 0} k_{i-2j}\lambda_{i-2j}, \quad \sum_{j} k_{i-2j} = m, \quad k_r \geq 0,$$

(we understand that $\lambda_r = 0$ if $r \leq 0$). This equivalence was established by Kleber [Kl] who developed an algorithm to study the combinatorics of the fermionic formula for an arbitrary simple Lie algebra $\mathfrak{g}$. Based on this algorithm, Kleber gave a description similar to the one above for the odd orthogonal and the symplectic Lie algebras. The exceptional cases were considered in [HKOTY] where they give formulas for the multiplicities for most nodes of the Dynkin diagram. It follows also from their work that the case of $N = 1$ is the crucial case, for they prove that this implies a weak fermionic formula, which they conjecture is equivalent to the fermionic formula.

Given this explicit description of the mulitplicities, it follows from the work of [C], [CP4] on minimal affinizations that, for any simple Lie algebra $\mathfrak{g}$, there exists up to $\mathbf{U}_q(\mathfrak{g})$–module isomorphisms, exactly one module $W_q^{\mathrm{aff}}(m\lambda_i) = \oplus m_\mu V_q(\mu)$ which can have the prescribed decomposition. This is the unique minimal affinization of $m\lambda_i$, which is characterized by the property: $m_\mu = 0$ *if $m\lambda_i - \mu$ is a non–negative linear combination of simple roots which lie in a Dynkin subdiagram of type A.* Thus, we need to understand the $\mathbf{U}_q(\mathfrak{g})$-decomposition of the minimal affinzations of $m\lambda_i$. We approach this problem as follows.

In [CP5], we showed that under natural conditions, the irreducible finite-dimensional representations of $\mathbf{U}_q(\hat{\mathfrak{g}})$ admit an integral form. This allows us to define the $q \to 1$ limit of these representations; these are finite-dimensional but generally *reducible* representations of the loop algebra of $\mathfrak{g}$. It follows by standard results that the decomposition of these representations of the loop algebra into a direct sum of irreducible representations of $\mathfrak{g}$ is the same as the decomposition in the quantum case. In section 1, we study the classical limit of the minimal affinizations and show that

for a classical simple Lie algebra $m_\mu \leq 1$ and that $m_\mu \neq 0$ implies that m_μ is given by the fermionic formula. In section 2, we work entirely in the quantum algebra to prove that $m_\mu = 1$ if μ is as given in (0.1). For this, we use a result proved in [K], [VV] which describes when a tensor product of fundamental representations of $\mathbf{U}_q(\hat{\mathfrak{g}})$ is cyclic.

Our methods also show the following for *any* finite-dimensional simple Lie algebra: *if a simple root α_i occurs with multiplicity one in the highest root of $\mathfrak{g}$, then the modules $V_q^{fin}(m\lambda_i)$ admit a structure of a $\mathbf{U}_q(\hat{\mathfrak{g}})$-module.* This was stated by Drinfeld in his work on Yangians, [Dr1]. We also can prove a generalization: *if a root α_i occurs with multiplicity 2 in the highest root, then the minimal affinization is multiplicity free as a $\mathbf{U}_q(\mathfrak{g})$-module.* In section 3, we summarize the results that our techniques prove for the exceptional algebras.

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1. THE CLASSICAL CASE

In this section, we study certain finite-dimensional modules for the loop algebra of $\mathfrak{g}$. These modules (see the discussion following Definition 1.2 for their definition) are the $q \to 1$ limit of irreducible representations of the quantum loop algebra, although this does not become clear until the conjecture of Kirillov and Reshetikhin is established. The main result of this section is Theorem 1.

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra of type X_n (where $X = A, B, C$ or D), let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and R the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $I = \{1, 2, \cdots, n\}$, fix a set of simple roots (resp. coroots) α_i (resp. h_i) $(i \in I)$, and let $R^+ \subset \mathfrak{h}^*$ be the corresponding set of positive roots. We assume that the simple roots are numbered as in [B]; in particular, the subset $\{j, j+1, \cdots, n\} \subset I$ defines a subalgebra of type X_{n-j+1}.

Let $Q = \bigoplus_{i=1}^n \mathbf{Z}\alpha_i$ (resp. $Q^+ = \bigoplus_{i=1}^n \mathbf{Z}^+\alpha_i$) denote the root (resp. positive root) lattice of $\mathfrak{g}$. For $\eta \in Q^+$, $\eta = \sum_i r_i\alpha_i$, we set $\operatorname{ht}\eta = \sum_i r_i$. Let P (resp. P^+) be the lattice of integral (resp. dominant integral) weights. For $i \in I$, let $\lambda_i \in P^+$ be the ith fundamental weight. Given $\mu = \sum_{r=1}^n k_r\lambda_r \in P^+$, set $\ell(\mu) = \sum_{r=1}^n k_r$.

Definition 1.1. For $i \in I$ and $m \in \mathbf{Z}^+$ define subsets $P(i, m)$ of P^+ as follows:

(i) If $\mathfrak{g}$ is of type A_n then $P(i, m) = \{m\lambda_i\}$ for all $i \in I$ and $m \in \mathbf{Z}^+$.

(ii) If $\mathfrak{g}$ is of type B_n, then

$$P(i, 1) = \{\lambda_i, \lambda_{i-2}, \cdots \lambda_0\}, \quad 1 \leq i < n,$$
$$P(n, 1) = \{\lambda_n\}, \quad P(n, 2) = \{2\lambda_n, \lambda_{n-2}, \lambda_{n-4}, \cdots, \lambda_0\},$$
$$P(i, m) = P(i, 1) + P(i, m-1), \quad 1 \leq i < n, \quad P(n, m) = P(n, m-2) + P(n, 2), \quad m \geq 3$$

where $\lambda_0 = 0$ if $i \in I$ is even and $\lambda_0 = \lambda_1$ if $i \in I$ is odd.

(iii) If $\mathfrak{g}$ is of type D_n, then set

$$P(i, 1) = \{\lambda_i, \lambda_{i-2}, \cdots \lambda_0\}, \quad 1 \leq i < n-1$$
$$P(i, m) = P(i, 1) + P(i, m-1), \quad 1 \leq i < n-1, n,$$

where $\lambda_0 = 0$ if $i \in I$ is even and $\lambda_0 = \lambda_1$ if $i \in I$ is odd. Set

$$P(i, m) = \{m\lambda_i\}, \quad i = n-1, n.$$

(iv) If $\mathfrak{g}$ is of type C_n, then,

$$P(i,1) = \lambda_i, \quad P(i,2) = \{2\lambda_i, 2\lambda_{i-1}, \cdots, 2\lambda_1, 0\}, \quad 1 \le i < n,$$
$$P(i,m) = P(i,m-2) + P(i,2), \quad m \ge 3, \quad 1 \le i < n,$$
$$P(n,m) = \{m\lambda_n\}.$$

The following lemma is trivially checked.

Lemma 1.1.   (i) *If $\mathfrak{g}$ is of type B_n and $1 \le i < n$, then,*

$$P(i,m) = \left\{ \sum_{j=0}^{[i/2]} k_{i-2j}\lambda_{i-2j} : \sum_j k_{i-2j} = m \right\},$$

$$P(n,m) = \left\{ \sum_{j=0}^{[i/2]} k_{i-2j}\lambda_{i-2j} : k_n + 2\sum_j k_{i-2j} = m \right\}.$$

(ii) *if $\mathfrak{g}$ is of type D_n, and $1 \le i \le n-2$, then*

$$P(i,m) = \left\{ \sum_{j=0}^{[i/2]} k_{i-2j}\lambda_{i-2j} : \sum_j k_{i-2j} = m \right\}.$$

(iii) *If $\mathfrak{g}$ is of type C_n, then we set $\lambda_0 = 0$ and*

$$P(i,m) = \left\{ \sum_{j=0}^{i} k_j\lambda_j : \sum_j k_j = m, \quad k_i \equiv m\,mod\,2, \quad k_j \equiv 0\,mod\,2, j \ne i \right\}.$$

$\square$

Let $\mathfrak{n}^{\pm}$ be the subalgebras

$$\mathfrak{n}^{\pm} = \bigoplus_{\pm\alpha \in R^+} \mathfrak{g}_\alpha.$$

Throughout this paper we shall (by abuse of notation) denote any non-zero element of $\mathfrak{g}_{\pm\alpha}$ as $x_\alpha^{\pm}$; of course, any two such elements are scalar multiples of each other, but for our purposes a precise choice of scalars is irrelevant. Thus, if $\alpha, \beta \in R^+$ is such that $\alpha \pm \beta \in R^+$, then we shall write

$$[x_\alpha^+, x_\beta^{\pm}] = x_{\alpha\pm\beta}^+,$$

etc.

For any Lie algebra $\mathfrak{a}$, the loop algebra of $\mathfrak{a}$ is the Lie algebra

$$L(\mathfrak{a}) = \mathfrak{a} \otimes \mathbf{C}[t, t^{-1}],$$

with commutator given by

$$[x \otimes t^r, y \otimes t^s] = [x,y] \otimes t^{r+s},$$

for $x, y \in \mathfrak{a}$, $r, s \in \mathbf{Z}$. For any $x \in \mathfrak{a}$, $m \in \mathbf{Z}$, we denote by x_m the element $x \otimes t^m \in L(\mathfrak{a})$. Let $\mathbf{U}(\mathfrak{a})$ be the universal enveloping algebra of $\mathfrak{a}$.

For $i \in I$, $k \in \mathbf{Z}$, define elements of $L(\mathfrak{g})$ by $e_i^{\pm} = x_{\alpha_i}^{\pm} \otimes 1$, $x_{i,k}^{\pm} = x_{\alpha_i}^{\pm} \otimes t^k$ and $e_0^{\pm} = x_{\theta_1}^{\mp} \otimes t^{\pm 1}$. Then, the elements $e_i^{\pm}$ $(i = 0, \cdots, n)$ generate $L(\mathfrak{g})$. We set

$$\mathbf{U}(L(\mathfrak{g})) = \mathbf{U}, \quad \mathbf{U}(\mathfrak{g}) = \mathbf{U}^{fin}.$$

We have

$$\mathbf{U} = \mathbf{U}(L(\mathfrak{n}^-))\mathbf{U}(L(\mathfrak{h}))\mathbf{U}(L(\mathfrak{n}^+)), \quad \mathbf{U}(\mathfrak{g}) = \mathbf{U}(\mathfrak{n}^-)\mathbf{U}(\mathfrak{h})\mathbf{U}(\mathfrak{n}^+).$$

Given $\lambda \in P^+$, let $V^{fin}(\lambda)$ be the unique irreducible finite-dimensional $\mathbf{U}^{fin}$-module with highest weight λ with highest weight vector v_λ. For all $\alpha \in R^+$, $h \in \mathfrak{h}$, we have

$$x_\alpha^+.v_\lambda = 0, \quad h.v = \lambda(h).v_\lambda, \quad (x_\alpha^-)^{\lambda(h_\alpha)+1}.v_\lambda = 0.$$

The action of $\mathfrak{g}$ on $V^{fin}(\lambda)$ extends to an action of $L(\mathfrak{g})$, by setting,

$$x_m.v = x.v, \quad \forall\, m \in \mathbf{Z}, \; x \in \mathfrak{g}.$$

We denote this $L(\mathfrak{g})$–module by $V(\lambda)$. For any finite-dimensional $\mathbf{U}^{fin}$-module V and any $\nu \in \mathfrak{h}^*$, let

$$V_\nu = \{v \in V : h.v = \nu(h)v \;\forall\, h \in \mathfrak{h}\}.$$

Since V is a direct sum of irreducible $\mathbf{U}^{fin}$-modules, we can write

$$V \cong \bigoplus_{\mu \in P^+} m_\mu(V)V^{fin}(\mu),$$

where $m_\mu(V) \geq 0$ is the multiplicity with which $V^{fin}(\mu)$ occurs in the sum.

We next recall the definition of certain highest weight modules, introduced in [CP5]; in fact, only the following special case will be needed. Let $\boldsymbol{\pi}_{i,m} = (\pi_1, \cdots, \pi_n)$ be the n-tuple of polynomials in $\mathbf{C}[u]$ given by

$$\pi_j(u) = 1 \;\; \text{if} \;\; j \neq i, \quad \pi_i(u) = (1-u)^m.$$

Definition 1.2. The $\mathbf{U}$-modules $W(\boldsymbol{\pi}_{i,m})$ are generated by an element $w_{i,m}$ subject to the relations

$$(1.1) \qquad x_{j,k}^+.w_{i,m} = 0, \quad h_k.w_{i,m} = m\lambda_i(h)w_{i,m} \quad (h \in \mathfrak{h}, \; k \in \mathbf{Z}),$$

$$(1.2) \qquad (x_{i,k}^-)^{m+1}.w_{i,m} = 0, \quad x_{j,k}^-.w_{i,m} = 0 \quad (j \neq i, \; k \in \mathbf{Z}).$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

The following proposition was proved in [CP5, Section2, Theorem 1].

Proposition 1.1. *The $\mathbf{U}$-module $W(\boldsymbol{\pi}_{i,m})$ is finite dimensional and*

$$\mathbf{U}(\mathfrak{n}^- \otimes \mathbf{C}[t]).w_{i,m} = W(\boldsymbol{\pi}_{i,m}).$$

Further, the module $V(m\lambda_i)$ is the unique irreducible quotient of $W(\boldsymbol{\pi}_{i,m})$. In particular, $m_{m\lambda_i}(W(\boldsymbol{\pi}_{i,m})) = 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

The elements

$$(1.3) \qquad\qquad\qquad x_{i,k}^-.w_{i,m} - x_{i,0}^-.w_{i,m} \qquad (k \in \mathbf{Z})$$

generate a proper $\mathbf{U}$-submodule of $W(\boldsymbol{\pi}_{i,m})$. Let $W(i,m)$ denote the quotient of $W(\boldsymbol{\pi}_{i,m})$ by this submodule. We continue to denote by $w_{i,m}$ the image of $w_{i,m} \in W(\boldsymbol{\pi}_{i,m})$ in $W_{i,m}$. The main result of this section is the following.

Theorem 1. *Let $i \in I$, $m \geq 0$. For all $\mu \in P^+$ we have $m_\mu(W(i,m)) \leq 1$. Further,*

$$m_\mu(W(i,m)) \neq 0 \implies \mu \in P(i,m).$$

The rest of the section is devoted to proving the theorem.

For $i \in I$ and $l = 0, 1, 2$, set

$$R(i,l) = \left\{ \sum_{k=1}^{n} m_k \alpha_k \in R^+ : m_i = l \right\}.$$

Clearly,

$$R^+ = \bigcup_{l=0}^{2} R(i,l).$$

For $i \in I$, $l = 0, 1, 2$, define the subspaces $\mathfrak{n}^{\pm}(i,l)$ in the obvious way. Then,

$$[\mathfrak{n}^{\pm}(i,l'), \mathfrak{n}^{\pm}(i,l)] = 0, \quad \text{if } l' + l > 2,$$
$$[\mathfrak{n}^{\pm}(i,l'), \mathfrak{n}^{\pm}(i,l)] = \mathfrak{n}^{\pm}(i,l'+l), \quad \text{if } l' + l \leq 2.$$

Proposition 1.2. *Let $\alpha \in R^+$, $f \in \mathbf{C}[t,t^{-1}]$. Then,*

$$\alpha \in R(i,l) \implies (x_{\alpha}^- \otimes f(t-1)^l).w_{i,m} = 0.$$

Proof. We proceed by induction on $\operatorname{ht} \alpha$. The case of $\operatorname{ht} \alpha = 1$ is clear from (1.2) and (1.3). Assume that the result holds for $\operatorname{ht} \alpha < r$. Choosing $j \in I$ so that $\beta = \alpha - \alpha_j \in R^+$, we get

$$x_{\alpha}^- \otimes fg = [x_{\alpha_j}^- \otimes f, x_{\beta}^- \otimes g],$$

for all $f, g \in \mathbf{C}[t,t^{-1}]$.

If $j \neq i$, then $\alpha, \beta \in R(i,l)$ for some $l = 0, 1, 2$. Now (1.2) gives

$$(x_{\alpha}^- \otimes f(t-a)^l).w_{i,m} = (x_{\alpha_j}^- x_{\beta}^- \otimes f(t-a)^l).w_{i,m}.$$

Since $\operatorname{ht} \beta < \operatorname{ht} \alpha$, the result follows. Assume now that $j = i$. If $\alpha \in R(i,1)$, then $\beta \in R(i,0)$ and we get by using induction and (1.3) that

$$(x_{\alpha}^- \otimes f(t-1)).w_{i,m} = -(x_{\beta}^- . x_{\alpha_i}^- \otimes f(t-1)).w_{i,m} = 0.$$

Finally, if $\alpha \in R(i,2)$, then $\beta \in R(i,1)$ and we have again by induction that

$$(x_{\alpha}^- \otimes f(t-1)^2).w = [x_{\alpha_i}^- \otimes (t-1), x_{\beta}^- \otimes f(t-1)].w_{i,m} = 0.$$

This proves the proposition. $\square$

The following is now immediate by applying the PBW theorem.

Corollary 1.1. *We have,*

$$W(i,m) = \mathbf{U}(\mathfrak{n}^-)\mathbf{U}(\mathfrak{n}^-(i,2) \otimes (t-1)).w_{i,m}.$$

In particular if $R(i,2) = \{\phi\}$ then

$$W(i,m) \cong V(m\lambda_i) \quad \forall \, m \in \mathbf{Z}_+.$$

$\square$

In view of this corollary, we can now assume that $\mathfrak{g}$ is of type B_n, C_n or D_n and that $i \neq 1$ (resp. $i \neq n$, $i \neq 1, n-1, n$). We list the sets $R(i, 2)$ explicitly in these cases. Define roots,

$$\theta_{l,k}^i = \sum_{j=l}^{k} \alpha_j + 2 \sum_{j=k+1}^{n} \alpha_j. \text{ if } \mathfrak{g} = B_n, \ \ 1 \leq l \leq k \leq i-1,$$

$$= \sum_{j=l}^{k} \alpha_j + 2 \left(\sum_{j=k+1}^{n-2} \alpha_j \right) + \alpha_{n-1} + \alpha_n, \text{ if } \mathfrak{g} = D_n, \ \ 1 \leq l \leq k \leq i-1,$$

$$= \sum_{j=l}^{k-1} \alpha_j + 2 \left(\sum_{j=k}^{n-1} \alpha_j \right) + \alpha_n, \text{ if } \mathfrak{g} = C_n, \ \ 1 \leq l \leq k \leq i.$$

The collection of all the $\theta_{k,l}^i$ is $R(i, 2)$. Let $\mathfrak{u}^i$ be the subalgebra of $\mathfrak{g}$ spanned by $\{x_{\theta_{j,j}^i}^- : 1 \leq j \leq i-1, \ \ i-1 \equiv j \mod 2\}$ (resp. $\{x_{\theta_{j,j}^i}^- : 1 \leq j \leq i\}$), if $\mathfrak{g}$ is of type B_n or D_n (resp. C_n).

To prove the next proposition only, we shall denote by $\mathfrak{g}_n$ the Lie algebra of type X_n and by $W_n(i, m)$ the module $W(i, m)$ etc. The assignment

$$x_{\alpha_j}^{\pm} \to x_{\alpha_{j+1}}^{\pm},$$

extends to an embedding of $\mathfrak{g}_{n-1} \to \mathfrak{g}_n$ and to the corresponding loop algebras. Let $\mathfrak{t}_n^i = \oplus_k \mathbf{C} x_{\theta_{1,k}^i}^-$ and let $\mathfrak{n}_{n-1}(i, 2)$ denote the image in $\mathfrak{n}_n$ of $\mathfrak{n}_{n-1}(i-1, 2)$ etc. Then,

$$\mathfrak{n}_n^-(i, 2) = \mathfrak{n}_{n-1}^-(i, 2) \oplus \mathfrak{t}_n^i \ \ \mathfrak{u}_n^i = \mathfrak{u}_{n-1}^i \oplus \mathbf{C} x_{\theta_{1,1}^i}^-.$$

Further, it is easy to see that there exists a $\mathbf{U}_{n-1}$–module map $W_{n-1}(i-1, m) \to W_n(i, m)$, for $i \in I_n$, $i > 1$ (and as stated earlier $i \neq n$ for C_n and $i \neq n-1, n$ for D_n) with image $\mathbf{U}_{n-1}.w_{i,m}$.

We now prove,

Proposition 1.3. *We have,*

$$W_n(i, m) = \mathbf{U}(\mathfrak{n}_n^-)\mathbf{U}(\mathfrak{u}_n^i \otimes (t-1)).w_{i,m}.$$

Proof. We prove this proposition by induction on n. In the case when $R(i, 2)$ consists of exactly one element, we have $\mathfrak{u}_n^i = \mathfrak{n}_n^-(i, 2)$ and the result is just Corollary 1.1. Hence the proposition is established for $B_2 = C_2$, for D_4 and for $i = 1$ for all C_n.

So to complete the inductive step, we can assume that $i > 1$ and that the result holds for $\mathfrak{g}_{n-1}$. Thus the induction hypothesis gives,

$$\mathbf{U}_{n-1}.w_{i,m} = \mathbf{U}(\mathfrak{n}_{n-1}^-)\mathbf{U}(\mathfrak{u}_{n-1}^i \otimes (t-1)).w_{i,m}$$

We now get,

$$W_n(i, m) = \mathbf{U}(\mathfrak{n}_n^-)\mathbf{U}(\mathfrak{t}_n^i \otimes (t-1))\mathbf{U}(\mathfrak{n}_{n-1}^-(i, 2) \otimes (t-1)).w_{i,m}$$

$$= \mathbf{U}(\mathfrak{n}_n^-)\mathbf{U}(\mathfrak{t}_n^i \otimes (t-1))\mathbf{U}(\mathfrak{n}_{n-1}^-)\mathbf{U}(\mathfrak{u}_{n-1}^i \otimes (t-1)).w_{i,m}.$$

Since $[\mathfrak{t}_n^i, \mathfrak{n}_n^-] \subset \mathfrak{t}_n^i$, we get

$$W_n(i, m) = \mathbf{U}(\mathfrak{n}_n^-)\mathbf{U}(\mathfrak{t}_n^i \otimes (t-1))\mathbf{U}(\mathfrak{u}_{n-1}^i \otimes (t-1)).w_{i,m}.$$

To complete the proof, we must show that

$$(1.4) \qquad \mathbf{U}(\mathfrak{t}_n^i \otimes (t-1))\mathbf{U}(\mathfrak{u}_{n-1}^i \otimes (t-1)).w_{i,m} \subset \mathbf{U}(\mathfrak{n}^-)\mathbf{U}(\mathfrak{u}_n^i \otimes (t-1)).w_{i,m}.$$

We do this in the case of D_n and when i is even, the proof in the other cases, is similar and simpler. Set $\theta_{l,k}^i = \theta_{l,k}$ and define elements $\gamma_j \in R^+$ by,

$$\gamma_j = \theta_{1,j} - \theta_{j,j} = \sum_{r=1}^{j-1} \alpha_r \quad \text{if } j \text{ is odd,}$$

$$\gamma_j = \theta_{1,j} - \theta_{j+1j+1} = \sum_{r=1}^{j+1} \alpha_r, \quad \text{if } j \text{ is even.}$$

Since i is even, we have $x_{\gamma_j}^-.w_{i,m} = 0$ for all $2 \le j \le i-1$. Now, a simple checking shows that

$$(x_{\gamma_2}^-)^{s_2}(x_{\gamma_3}^-)^{s_3} \cdots (x_{\gamma_{i-1}}^-)^{s_{i-1}}$$
$$\times (x_{\theta_{1,1}}^- \otimes (t-1))^{r_1}(x_{\theta_{3,3}}^- \otimes (t-1))^{r_2+r_3+s_3} \cdots (x_{\theta_{i-1,i-1}}^- \otimes (t-1))^{r_{i-2}+r_{i-1}+s_{i-1}}.w_{i,m}$$
$$= (x_{\theta_{1,1}}^- \otimes (t-1))^{r_1}[(x_{\gamma_2}^-)^{r_2}(x_{\gamma_3}^-)^{r_3}, (x_{\theta_{3,3}}^- \otimes (t-1))^{r_2+r_3+s_3}] \cdots$$
$$\times [(x_{\gamma_{i-2}}^-)^{r_{i-2}}(x_{\gamma_{i-1}}^-)^{r_{i-1}}, x_{\theta_{i-1,i-1}}^- \otimes (t-1))^{r_{i-2}+r_{i-1}+s_{i-1}}].w_{i,m}$$
$$= (x_{\theta_{1,1}}^- \otimes (t-1))^{r_1}(x_{\theta_{1,2}}^- \otimes (t-1))^{r_2} \cdots (x_{\theta_{1,i-1}}^- \otimes (t-1))^{r_{i-1}}$$
$$\times (x_{\theta_{3,3}}^- \otimes (t-1))^{s_3}(x_{\theta_{5,5}}^- \otimes (t-1))^{s_5} \cdots (x_{\theta_{i-1,i-1}}^- \otimes (t-1))^{s_{i-1}}.w_{i,m},$$

where the last equality follows from the definition of the γ_j's and noting that $\theta_{j,j} + \gamma_k + \gamma_l \notin R^+$. This clearly proves (1.4) and the proof of the proposition is complete. $\qquad\square$

Proof of Theorem 1. Set $l = \dim \mathfrak{u}^i$ and let $\le$ be the lexicographic ordering on $\mathbf{Z}_+^l$. Given $\mathbf{s} \in \mathbf{Z}_+^l$, let

$$\mathbf{x_s} = \prod_{j=1}^{i-1} (x_{\theta_{j,j}}^- \otimes (t-1))^{s_j},$$

if $\mathfrak{g}$ is of type C_n, the coresponding analogues for B_n and D_n are defined in the obvious way.

Let $W_{\mathbf{o}}$ be the $\mathfrak{g}$–submodule of $W(i,m)$ generated by $w_{i,m}$ and let W_1 be a $\mathfrak{g}$–module such that

$$W(i,m) = W_{\mathbf{o}} \oplus W_1.$$

If $W_1 \ne 0$, choose $\mathbf{s}_1$ minimal so that the element $\mathbf{x}_{\mathbf{s}_1}.w_{i,m}$ has a non–zero projection $w_{\mathbf{s}_1}$ onto W_1. Now choose a $\mathfrak{g}$–submodule W_2 of W_1 so that,

$$W_1 = \mathbf{U}(\mathfrak{g}).w_{\mathbf{s}_1} \oplus W_2.$$

Repeating, we see that we can find a finite number of elements, say $\{w_{\mathbf{s}_j} : 1 \le j \le k\}$, with $\mathbf{s}_1 < \mathbf{s}_2 < \cdots < \mathbf{s}_k$ such that

$$W(i,m) = W_{\mathbf{o}} \oplus W_{\mathbf{s}_1} \oplus \cdots \oplus W_{\mathbf{s}_k},$$

where $W_{\mathbf{s}_j} = \mathbf{U}(\mathfrak{g}).w_{\mathbf{s}_j}$. Notice that by choice, the projection of $\mathbf{x}_{\mathbf{s}}.w_{i,m}$ onto $W_{\mathbf{s}_j}$ is zero if $\mathbf{s} < \mathbf{s}_j$. We claim that,

$$(1.5) \qquad\qquad\qquad x_\alpha^+.w_{\mathbf{s}_j} = 0, \quad \forall \ \alpha \in R^+.$$

From now on, we assume that $\mathfrak{g}$ is of type C_n, the proof in the other cases is similar. Thus, notice that if $k \neq i$, we have

$$x_{\alpha_k}^+ . \mathbf{x}_{\mathbf{s}_j} . w_{i,m} = 0 \ \text{ if } \ k > i,$$
$$= x_{\theta_{k,k}-\alpha_k}^- \otimes (t-1)(x_{\theta_{k,k}}^- \otimes (t-1))^{s_k-1} \prod_{j' \neq k} (x_{\theta_{j',j'}}^- \otimes (t-1))^{s_{j'}} . w_{i,m},$$
$$= x_{\alpha_k}^- (x_{\theta_{k,k}}^- \otimes (t-1))^{s_k-1} (x_{\theta_{k+1,k+1}}^- \otimes (t-1))^{s_{k+1}+1} \prod_{j' \neq k,k+1} (x_{\theta_{j',j'}}^- \otimes (t-1))^{s_{j'}} . w_{i,m}.$$

But the right hand side of the last equality is clearly in $\oplus W_{\mathbf{s}_r}$ with $\mathbf{s}_r < \mathbf{s}_j$. This gives (1.5) if $k \neq i$. If $k = i$, then, we have,

$$x_{\alpha_i}^+ . \mathbf{x}_{\mathbf{s}_j} . w_{i,m} = x_{\theta_{i,i}-\alpha_i}^- \otimes (t-1)(x_{\theta_{k,k}}^- \otimes (t-1))^{s_k-1} \prod_{j' \neq k} (x_{\theta_{j',j'}}^- \otimes (t-1))^{s_{j'}} . w_{i,m},$$
$$= 0,$$

where the last equality follows from the fact that $\theta_{i,i} - \alpha_i \in R(i,1)$. This proves (1.5) completely and and hence we get that if $m_\mu(W(i,m)) \neq 0$ then $\mu = m\lambda_i$ or μ is the weight of the element $w_{\mathbf{s}_j}$ for some j.

A simple calculation shows that $\theta_{j,j} = 2\lambda_j - 2\lambda_{j-1}$ and hence the weight of the element $w_{\mathbf{s}_j}$ where $\mathbf{s}_j = (s_{j1}, s_{j2}, \cdots s_{jl})$ is

$$\mu_j = (m - 2s_{ji})\lambda_i + 2(s_{ji} - s_{ji-1})\lambda_{i-1} + \cdots + 2(s_{j2} - s_{j1})\lambda_1.$$

Since μ_j must be a dominant integral weight we see that $\mu_j \in P(i,m)$. Further, the μ_j are clearly distinct and hence Theorem 1 is proved. $\qquad \square$

2. The quantum case

In this section we recall the definition of the quantum affine algebras and several results on the irreducible finite-dimensional representations of $\mathbf{U}_q(\hat{\mathfrak{g}})$. We then define the module whose decomposition we are interested in and establish the Kirillov-Reshetikhin conjecture in this case. We continue to assume that $\mathfrak{g}$ is of type X_n, where $X = A, B, C$ or D.

Let q be an indeterminate, let $\mathbf{C}(q)$ be the field of rational functions in q with complex coefficients, and let $\mathbf{A} = \mathbf{C}[q, q^{-1}]$ be the subring of Laurent polynomials. For $r, m \in \mathbf{N}$, $m \geq r$, define

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [m]! = [m][m-1]\ldots[2][1], \quad \begin{bmatrix} m \\ r \end{bmatrix} = \frac{[m]!}{[r]![m-r]!}.$$

Then, $\begin{bmatrix} m \\ r \end{bmatrix} \in \mathbf{A}$.

We now recall the definition of the quantum affine algebra. Let $\hat{A} = (a_{ij})$ be the $(n+1) \times (n+1)$ extended Cartan matrix associated to $\mathfrak{g}$. Let $\hat{I} = I \cup \{0\}$. Fix non–negative integers d_i, $i \in \hat{I}$ such that the matix $(d_i a_{ij})$ is symmetric. Set $q_i = q^{d_i}$ and $[m]_i = [m]_{q_i}$.

Proposition 2.1. *There is a Hopf algebra* $\tilde{\mathbf{U}}_q$ *over* $\mathbf{Q}(q)$ *which is generated as an algebra by elements* E_{α_i}, F_{α_i}, $K_i^{\pm 1}$ $(i \in \hat{I})$, *with the following defining relations:*

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \qquad K_i K_j = K_j K_i,$$

$$K_i E_{\alpha_j} K_i^{-1} = q_i^{a_{ij}} E_{\alpha_j},$$

$$K_i F_{\alpha_j} K_i^{-1} = q_i^{-a_{ij}} F_{\alpha_j},$$

$$[E_{\alpha_i}, F_{\alpha_j}] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_i (E_{\alpha_i})^r E_{\alpha_j} (E_{\alpha_i})^{1-a_{ij}-r} = 0 \qquad \text{if } i \neq j,$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_i (F_{\alpha_i})^r F_{\alpha_j} (F_{\alpha_i})^{1-a_{ij}-r} = 0 \qquad \text{if } i \neq j.$$

The comultiplication of $\tilde{\mathbf{U}}_q$ *is given on generators by*

$$\Delta(E_{\alpha_i}) = E_{\alpha_i} \otimes 1 + K_i \otimes E_{\alpha_i}, \quad \Delta(F_{\alpha_i}) = F_{\alpha_i} \otimes K_i^{-1} + 1 \otimes F_{\alpha_i}, \quad \Delta(K_i) = K_i \otimes K_i,$$

for $i \in \hat{I}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Set $K_\theta = \prod_{i=1}^n K_i^{r_i/d_i}$, where $\theta = \sum r_i \alpha_i$ is the highest root in R^+. Let $\mathbf{U}_q$ be the quotient of $\tilde{\mathbf{U}}_q$ by the ideal generated by the central element $K_0 K_\theta^{-1}$; we call this the quantum loop algebra of $\mathfrak{g}$.

It follows from [Dr2], [B], [J] that $\mathbf{U}_q$ is isomorphic to the algebra with generators $\mathbf{x}_{i,r}^{\pm}$ $(i \in I, r \in \mathbf{Z})$, $K_i^{\pm 1}$ $(i \in I)$, $\mathbf{h}_{i,r}$ $(i \in I, r \in \mathbf{Z}\backslash\{0\})$ and the following defining relations:

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i,$$

$$K_i \mathbf{h}_{j,r} = \mathbf{h}_{j,r} K_i,$$

$$K_i \mathbf{x}_{j,r}^{\pm} K_i^{-1} = q_i^{\pm a_{ij}} \mathbf{x}_{j,r}^{\pm},$$

$$[\mathbf{h}_{i,r}, \mathbf{h}_{j,s}] = 0, \quad [\mathbf{h}_{i,r}, \mathbf{x}_{j,s}^{\pm}] = \pm \frac{1}{r} [r a_{ij}]_i \mathbf{x}_{j,r+s}^{\pm},$$

$$\mathbf{x}_{i,r+1}^{\pm} \mathbf{x}_{j,s}^{\pm} - q_i^{\pm a_{ij}} \mathbf{x}_{j,s}^{\pm} \mathbf{x}_{i,r+1}^{\pm} = q_i^{\pm a_{ij}} \mathbf{x}_{i,r}^{\pm} \mathbf{x}_{j,s+1}^{\pm} - \mathbf{x}_{j,s+1}^{\pm} \mathbf{x}_{i,r}^{\pm},$$

$$[\mathbf{x}_{i,r}^+, \mathbf{x}_{j,s}^-] = \delta_{i,j} \frac{\psi_{i,r+s}^+ - \psi_{i,r+s}^-}{q_i - q_i^{-1}},$$

$$\sum_{\pi \in \Sigma_m} \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_i \mathbf{x}_{i,r_{\pi(1)}}^{\pm} \cdots \mathbf{x}_{i,r_{\pi(k)}}^{\pm} \mathbf{x}_{j,s}^{\pm} \mathbf{x}_{i,r_{\pi(k+1)}}^{\pm} \cdots \mathbf{x}_{i,r_{\pi(m)}}^{\pm} = 0, \quad \text{if } i \neq j,$$

for all sequences of integers $r_1, \ldots, r_m$, where $m = 1 - a_{ij}$, Σ_m is the symmetric group on m letters, and the $\psi_{i,r}^{\pm}$ are determined by equating powers of u in the formal power series

$$\sum_{r=0}^\infty \psi_{i,\pm r}^{\pm} u^{\pm r} = K_i^{\pm 1} \exp\left(\pm (q_i - q_i^{-1}) \sum_{s=1}^\infty \mathbf{h}_{i,\pm s} u^{\pm s} \right).$$

For $i \in I$, the above isomorphism maps E_{α_i} to $\mathbf{x}_{i,0}^+$ and F_{α_i} to $\mathbf{x}_{i,0}^-$. The sub-algebra generated by E_{α_i}, F_{α_i}, $i \in I$, is the quantized enveloping algebra $\mathbf{U}_q^{fin}$ associated to $\mathfrak{g}$,

Define the q-divided powers

$$(\mathbf{x}_{i,k}^\pm)^{(r)} = \frac{(\mathbf{x}_{i,k}^\pm)^r}{[r]_i!},$$

for all $i \in I$, $k \in \mathbf{Z}$, $r \geq 0$. The elements $E_{\alpha_i}^{(r)}$ etc. are defined similarly. Let $\mathbf{U_A}$ be the $\mathbf{A}$-subalgebra of $\mathbf{U}_q$ generated by the $K_i^{\pm 1}$, $(\mathbf{x}_{i,k}^\pm)^{(r)}$ ($i \in I$, $k \in \mathbf{Z}$, $r \geq 0$).

Lemma 2.1. *The subalgebra $\mathbf{U_A}$ is an $\mathbf{A}$–lattice in $\mathbf{U}_q$, and*

$$\mathbf{U}_q = \mathbf{C}(q) \otimes_{\mathbf{A}} \mathbf{U_A}.$$

Proof. Let $\tilde{\mathbf{U}}_{\mathbf{A}}$ be the $\mathbf{A}$–subalgebra generated by the elements $E_{\alpha_i}^{(r)}$, $F_{\alpha_i}^{(r)}$, $i \in \hat{I}$. It is proved in [L2] that $\tilde{\mathbf{U}}_{\mathbf{A}}$ is an $\mathbf{A}$–lattice and that

$$\mathbf{U}_q = \mathbf{C}(q) \otimes_{\mathbf{A}} \tilde{\mathbf{U}}_{\mathbf{A}}.$$

Hence to prove the lemma it suffices to show that the $\mathbf{U_A} = \tilde{\mathbf{U}}_{\mathbf{A}}$. For this, in view of the isomorphism between the two presentations it suffices to show that the elements $E_{\alpha_0}^{(r)}$ and $F_{\alpha_0}^{(r)}$ are in $\mathbf{U_A}$. In the simply laced case this was proved in [BCP, Proposition 2.6]. The proof given there works as long as there exists a simple root α_{i_0} which occurs with mulitplicity one in θ, i.e $r_{i_0} = 1$. An inspection shows that this is true for the classical simple Lie algebras. □

Given $i, j \in I$ with $a_{ij} = -2$ and $k, l \in \mathbf{Z}$, it is easy to see that the subalgebra generated by the elements $\mathbf{x}_{i,k}^\pm$ and $\mathbf{x}_{j,l}^\pm$ is isomorphic to the quantized enveloping algebra of $\mathbf{U}_q(sp_5)$. Define elements,

$$\gamma_{k,l}(q) = \mathbf{x}_{i,k}^- \mathbf{x}_{j,l}^- - q^2 \mathbf{x}_{j,l}^- \mathbf{x}_{i,k}^-, \quad (\gamma_{k,l}(q))^{(r)} = \frac{(\gamma_{k,l}(q))^r}{[r]_i!},$$

and

$$\gamma'_{i,k}(q) = [\mathbf{x}_{i,l}^-, \gamma_{k,l}(q)], \quad (\gamma'_{k,l}(q))^{(r)} = \frac{(\gamma'_{k,l}(q))^r}{[r]_j!}.$$

It is easy to see using the defining relations in $\mathbf{U}_q$ that,

$$\gamma_{k,l}(q) = q^2 \gamma_{k-1,l+1}(q^{-1}), \quad \gamma'_{k,l}(q) = q^2 \gamma'_{k-1,l+1}(q^{-1}).$$

Lemma 2.2. *Assume that $i, j \in I$ is such that $a_{ij} = -2$. Then,*

$$(\mathbf{x}_{i,k}^-)^{(a)}(\mathbf{x}_{j,l}^-)^{(b)} = \sum_{r,t \in \mathbf{Z}_+} f_{r,t}(\mathbf{x}_{j,l}^-)^{(b-r-t)}(\gamma_{k,l}(q))^{(r)}(\gamma'_{k,l}(q))^{(t)}(\mathbf{x}_{i,k}^-)^{(a-r-2t)},$$

where $f_{r,t} \in q^{\mathbf{Z}_+}$. In particular the elements $(\gamma_{k,l}(q^{\pm 1}))^{(r)}$ and $(\gamma'_{k,l}(q^{\pm 1}))^{(r)}$ are in $\mathbf{U_A}$.

Proof. This follows from the result proved in [L2] for the quantized enveloping algebra of sp_5. □

For any $\mathbf{U}_q^{fin}$-module V_q and any $\mu \in P$, set

$$(V_q)_\mu = \{v \in V_q : K_i.v = q_i^{\mu(h_i)}v, \ \ \forall \ i \in I\}.$$

We say that V_q is a module of type 1 if

$$V_q = \bigoplus_{\mu \in P}(V_q)_\mu.$$

From now on, we shall only be working with $\mathbf{U}_q^{fin}$-modules of type 1.

The irreducible finite-dimensional $\mathbf{U}_q^{fin}$-modules are parametrized by P^+. Thus, for each $\lambda \in P^+$, there exists a unique irreducible finite-dimensional module $V_q^{fin}(\lambda)$ generated by a non-zero element v_λ, with defining relations

$$\mathbf{x}_{i,0}^+.v_\lambda = 0, \ \ K_i.v_\lambda = q^{\lambda(h_i)}v_\lambda, \ \ (\mathbf{x}_{i,0}^-)^{\lambda(h_i)+1}.v_\lambda = 0, \ \ \forall \ i \in I.$$

Further,

$$(V_q^{fin}(\lambda))_\mu \neq 0 \implies \mu \in \lambda - Q^+.$$

Set $V_{\mathbf{A}}^{fin}(\lambda) = \mathbf{U}_{\mathbf{A}}.v_\lambda$. Then,

$$V_q^{fin}(\lambda) = \mathbf{C}(q) \otimes_{\mathbf{A}} V_{\mathbf{A}}^{fin}(\lambda),$$

and

$$\overline{V_q^{fin}(\lambda)} = \mathbf{C}_1 \otimes_{\mathbf{A}} V_{\mathbf{A}}^{fin}(\lambda).$$

Then [L1], $\overline{V_q^{fin}(\lambda)}$ is a module for $\mathbf{U}$ and is isomorphic to $V^{fin}(\lambda)$. It is also known [L1] that any finite-dimensional $\mathbf{U}_q^{fin}$-module V_q is a direct sum of irreducible modules; we let $m_\mu(V_q)$ be the multiplicity with which $V_q^{fin}(\mu)$ occurs in V_q.

The type 1 irreducible finite–dimensional $\mathbf{U}_q$-modules are parametrized by n-tuples of polynomials $\boldsymbol{\pi}_q = (\pi_1(u), \cdots, \pi_n(u))$, where the $\pi_r(u)$ have coefficients in $\mathbf{C}(q)$ and constant term 1. Let us denote the corresponding module by $V_q(\boldsymbol{\pi}_q)$. Then, [CP3], there exists a unique (up to scalars) element $v_{\boldsymbol{\pi}_q} \in V_q(\boldsymbol{\pi}_q)$ satisfying

$$(2.1) \qquad \mathbf{x}_{k,r}^+.v_{\boldsymbol{\pi}_q} = 0, \ \ K_i.v_{\boldsymbol{\pi}_q} = q^{\deg \pi_i}v_{\boldsymbol{\pi}_q},$$

and

$$(2.2) \qquad \mathbf{h}_{i,k}.v_{\boldsymbol{\pi}_q} = d_{i,k}.v_{\boldsymbol{\pi}_q}, \ \ (\mathbf{x}_{i,k}^-)^{\deg \pi_i+1}.v_{\boldsymbol{\pi}_q} = 0,$$

where the $d_{i,k}$ are determined from the functional equation

$$\exp\left(-\sum_{k \geq 0} \frac{d_{i,\pm k}u^k}{k}\right) = \pi_i^\pm(u),$$

where $\pi_i^+(u) = \pi_i(u)$ and $\pi_i^-(u) = u^{\deg \pi_i}\pi_i(u^{-1})/\left(u^{\deg \pi_i}\pi_i(u^{-1})\big|_{u=0}\right)$. We remark that these are in general **not** the defining relations of $V_q(\boldsymbol{\pi}_q)$. Set,

$$V_{\mathbf{A}}(\boldsymbol{\pi}_q) = \mathbf{U}_{\mathbf{A}}.v_{\boldsymbol{\pi}_q}.$$

Proposition 2.2. *Suppose that the n–tuple $\boldsymbol{\pi}_q = (\pi_1, \cdots \pi_n)$ is such that for all $j \in I$, $\pi_j(u) \in \mathbf{A}[u]$. Regarded as an $\mathbf{A}$–module $V_{\mathbf{A}}(\boldsymbol{\pi}_q)$ is free of rank equal to $dim_{\mathbf{C}(q)}V_q(\boldsymbol{\pi}_q)$.*

Proof. In the simply–laced case, this was proved in [CP5, Proposition 4.4]. The argument given there can be extended to include the case of B_n and C_n as follows. The crucial step is to prove that an element of the form

$$(\mathbf{x}_{i_1,k_1}^-)^{(s_1)}(\mathbf{x}_{i_2,k_2}^-)^{(s_2)}\cdots(\mathbf{x}_{i_l,k_l}^-)^{(s_l)}.v_{\boldsymbol{\pi}_q},$$

can be rewritten as an $\mathbf{A}$–linear combination of elements

$$(\mathbf{x}_{i_1',k_1'}^-)^{(s_1')}(\mathbf{x}_{i_2',k_2'}^-)^{(s_2')}\cdots(\mathbf{x}_{i_l',k_l'}^-)^{(s_l')}.v_{\boldsymbol{\pi}_q}, \quad 0 \leq k_j' \leq N(\eta)$$

where $N(\eta)$ depends only on $\eta = \sum_j s_j\alpha_{i_j}$ and $\boldsymbol{\pi}_q$. The proof proceeds by an induction on ht η, The case $\eta = s\alpha_i$ was done in [CP5]. So we can assume that $s_1 \neq 0$ and $s_2 \neq 0$ and that $k_j \leq N(\eta - s_1\alpha_{i_1})$ for all $2 \leq j \leq l$. If $a_{i_1,i_2} = 0$ the result is obvious. If $a_{i_1,i_2} = -1$ then the inductive step is proved in [CP5].

It remains to prove the inductive step when $a_{i_1,i_2} = -2$. We assume $k_1 \geq 0$, (the case $k_1 < 0$ is similar, see [CP5]) and proceed by induction on k_1, with induction beginning at $k_1 = N(\eta - s_1\alpha_1)$. By Lemma 2, we see that the elements $(\gamma_{k_1,k_2})^{(r)}$ and $(\gamma_{k_1,k_2}')^{(t)}$ belong to the $\mathbf{U_A}$ subalgebra generated by the elements $\{(\mathbf{x}_{i,m}^-)^{(s)} : i \in I, s \in \mathbf{Z}^+, 0 \leq m \leq N(\eta - s_1\alpha_1) + 2\}$. Now using Lemma 2.2 and the induction hypothesis we see that the element $(\mathbf{x}_{i_1,k_1}^-)^{(s_1)}(\mathbf{x}_{i_2,k_2}^-)^{(s_2)}\cdots(\mathbf{x}_{i_l,k_l}^-)^{(s_l)}.v_{\boldsymbol{\pi}_q}$ can be rewritten as a linear combination of similar elements but with the $k_j \leq N(\eta - s_1\alpha_1) + 2$ for all j thus completing the inductive step.

To complete the proof of the proposition, observe that since the module is finite–dimensional over $\mathbf{C}(q)$,

$$(\mathbf{x}_{i_1,k_1}^-)^{(s_1)}(\mathbf{x}_{i_2,k_2}^-)^{(s_2)}\cdots(\mathbf{x}_{i_l,k_l}^-)^{(s_l)}.v_{\boldsymbol{\pi}_q} = 0$$

for all $l >> 0$ and for all but finitely many values of $s_1, s_2, \cdots s_l$. It follows now that, there exists an integer $N \geq 0$ such that the elements

$$(\mathbf{x}_{i_1,k_1}^-)^{(s_1)}(\mathbf{x}_{i_2,k_2}^-)^{(s_2)}\cdots(\mathbf{x}_{i_l,k_l}^-)^{(s_l)}.v_{\boldsymbol{\pi}_q}, \quad 0 \leq k_j < N$$

span $\mathbf{U_A}.v_{\boldsymbol{\pi}_q}$. This means that $V_{\mathbf{A}}(\boldsymbol{\pi}_q)$ is a finitely generated $\mathbf{A}$–module and hence is a free $\mathbf{A}$ module. Since these elements also clearly span $V_q(\boldsymbol{\pi}_q)$ over $\mathbf{C}(q)$, the proposition follows. $\quad\square$

Given, $\boldsymbol{\pi} = (\pi_1, \pi_2, \cdots, \pi_n)$ such that $\pi_j(u) \in \mathbf{A}[u]$ for all $j \in I$, set

$$(2.3) \qquad\qquad\qquad \overline{V_q(\boldsymbol{\pi}_q)} = \mathbf{C}_1 \otimes_{\mathbf{A}} V_{\mathbf{A}}(\boldsymbol{\pi}_q).$$

Let $\overline{\boldsymbol{\pi}_q}$ be the n-tuple of polynomials with coefficients in $\mathbf{C}$ obtained by setting $q = 1$ in the components of $\boldsymbol{\pi}_q$. Then, $\overline{V_q(\boldsymbol{\pi}_q)}$ is a $\mathbf{U}$-module generated by $1 \otimes v_{\boldsymbol{\pi}_q}$ and satisfying the relations in (2.1) and (2.2) with the generators $\mathbf{x}_{i,k}^{\pm}$ etc. being replaced by their classical analogues. Further, if we write

$$V_q(\boldsymbol{\pi}_q) = \bigoplus_{\mu \in P^+} m_\mu(V_q(\boldsymbol{\pi}_q))V_q^{fin}(\mu),$$

as $\mathbf{U}_q^{fin}$-modules, then

$$\overline{V_q(\boldsymbol{\pi}_q)} = \bigoplus_{\mu \in P^+} m_\mu(V_q(\boldsymbol{\pi}_q))V^{fin}(\mu),$$

as $\mathbf{U}$-modules.

From now on, we shall only be interested in the following case. Thus, for $i \in I$, $m \geq 0$, $a \in \mathbf{C}^\times$, let $\boldsymbol{\pi}_q(i, m, a)$ be the n-tuple of polynomials given by

$$\pi_j(u) = 1, \quad \text{if } j \neq i,$$
$$\pi_i(u) = (1 - au)(1 - aq^{-2}u) \cdots (1 - aq^{-2m+2}u).$$

We denote the corresponding $\mathbf{U}_q$-module by $V_q(i, m, a)$. In the case when $a = 1$, we set $V_q(i, m, 1) = V_q(i, m)$. For all $a \in \mathbf{C}^\times$ we let $v_{i,m}$ denote the vector $v_{\boldsymbol{\pi}_q(i,m,a)}$.

Given any connected subset $J \subset I$, let $\mathbf{U}_q^J$ be the quantized enveloping algebra of $L(\mathfrak{g}_J)$, this clearly maps to the subalgebra of $\mathbf{U}_q$ generated by the elements $\{\mathbf{x}_{j,k}^\pm : j \in J, k \in \mathbf{Z}\}$.

Lemma 2.3. *Let $J = \{i\}$, $m \geq 0$. Then $\mathbf{U}_{J,q}.v_{i,m} \subset V_q(i, m)$ is an irreducible $\mathbf{U}_{J,q}$–module and*

$$\mathbf{x}_{i,k}^-.v_{i,m} = q^k \mathbf{x}_{i,0}^-.v_{i,m}.$$

In particular,

$$dim_{\mathbf{C}(q)}(V_q(i, m))_{m\lambda_i - \alpha_i} = 1.$$

Proof. It is easy to see that $\mathbf{U}_{J,q}.w_{i,m}$ is an irreducible $\mathbf{U}_{J_q}$–module. Further, a simple checking shows that the elements $\{\mathbf{x}_{i,k}^-.v_{i,m} - q^k \mathbf{x}_i^-.v_{i,m} : k \in \mathbf{Z}\}$ generate a submodule of $V_q(i, m)$ not containing $v_{i,m}$ and hence must be zero. $\square$

In view of (2.3) it follows from the preceding lemma, that

$$\dim \overline{(V_q(i, m))}_{m\lambda_i - \alpha_i} = 1.$$

The next lemma is immediate.

Lemma 2.4. *The $\mathbf{U}$-module $\overline{V_q(i, m)}$ is a quotient of $W(i, m)$.* $\square$

It now follows from Theorem 1 that

Lemma 2.5. *For all $\mu \in P^+$ we have $m_\mu(V_q(i, m)) \leq 1$. Further,*
$$m_\mu(V_q(i, m)) \neq 0 \implies \mu \in P(i, m).$$

$\square$

The main result of this paper is

Theorem 2. *Let $\mu \in P^+$. Then, $m_\mu(V_q(i, m)) \leq 1$ and $m_\mu(V_q(i, m)) \neq 0$ if and only if $\mu \in P(i, m)$.*

The following corollary is immediate.

Corollary 2.1. *For all $i \in I$ and $m \geq 0$, we have*
$$W(i, m) \cong \overline{V_q(i, m)}.$$

$\square$

In view of Lemma 2.5, to prove Theorem 2, it suffices to prove

Proposition 2.3. *Let $\mu \in P(i, m)$. Then, $m_\mu(V_q(i, m)) = 1$.*

The rest of the section is devoted to proving this result. Observe that when $P(i, m) = \{m\lambda_i\}$ there is nothing to prove. This means that we can assume $\mathfrak{g}$ is of type B, C or D. We shall need the following result which is a special case of a theorem of [K], (see [VV] for a different proof in the simply–laced case).

Proposition 2.4. *For all $i \in I$, $m \in \mathbf{Z}^+$, the $\mathbf{U}_q$–module $V_q(\lambda_i, 1) \otimes V_q(\lambda_i, q^{-2}) \otimes \cdots \otimes V_q(\lambda_i, q^{-2m+2})$ is generated by $v_{i,1} \otimes v_{i,1} \otimes \cdots \otimes v_{i,1}$.* $\square$

Given two n-tuples of polynomials $\boldsymbol{\pi}_q$ and $\tilde{\boldsymbol{\pi}}_q$, let

$$\boldsymbol{\pi}_q \, \tilde{\boldsymbol{\pi}}_q = (\pi_1 \tilde{\pi}_1, \ldots, \pi_n \tilde{\pi}_n).$$

Lemma 2.6. *The assignment $v_{i,1} \otimes v_{i,1} \otimes \cdots \otimes v_{i,1} \mapsto v_{i,m}$ extends to a surjective homomorphism of $\mathbf{U}_q$–modules $phi_m^i : V_q(\lambda_i, 1) \otimes V_q(\lambda_i, q^{-2}) \otimes \cdots \otimes V_q(\lambda_i, q^{-2m+2}) \rightarrow V_q(i, m)$.*

Proof. It was proved in [CP3], [Da] that

$$\Delta(\mathbf{h}_{i,k}) = \mathbf{h}_{i,k} \otimes 1 + 1 \otimes \mathbf{h}_{i,k} \quad \mod \mathbf{U} \otimes \mathbf{U}\mathbf{U}(>)_+,$$

where $\mathbf{U}(>)$ is the subalgebra generated by $\mathbf{x}_{j,l}^+$ for all $j \in I$ and $l \in \mathbf{Z}_+$, and $\mathbf{U}(>)_+$ is the augmentation ideal. It is now easy to see using (2.1) and (2.2) that the action of

$$\mathbf{h}_{i,k}.v_{i,1} \otimes v_{i,1} \otimes \cdots \otimes v_{i,1} = (-1)^k q_i^{\binom{m}{k} t} \begin{bmatrix} m \\ k \end{bmatrix}_{q_i} v_{i,1} \otimes v_{i,1} \otimes \cdots \otimes v_{i,1},$$

and

$$\mathbf{x}_{i,k}^+.v_{i,1} \otimes v_{i,1} \otimes \cdots \otimes v_{i,1} = 0,$$

for all $i \in I$ and $k \in \mathbf{Z}$. This proves the lemma. $\square$

To prove Proposition 2.3, we proceed by induction on m. We first show that induction starts.

Lemma 2.7.

(i) *Assume $\mathfrak{g}$ is of type B_n. If $i \neq n$, then as $\mathbf{U}_q^{fin}$–modules we have*

$$V_q(i, 1) \cong \bigoplus_{j=0}^{[i/2]} V_q^{fin}(\lambda_{i-2j}), \quad i \neq n,$$

$$V_q(n, 1) \cong V_q^{fin}(\lambda_n),$$

$$V_q(n, 2) \cong V_q^{fin}(2\lambda_n) \bigoplus_{j=0}^{[n/2]} V_q^{fin}(\lambda_{n-2j}).$$

(ii) *Assume that $\mathfrak{g}$ is of type D_n and that $1 \leq i \leq n - 2$. Then, as $\mathbf{U}_q^{fin}$-modules,*

$$V_q(i, 1) \cong \bigoplus_{j=0}^{[i/2]} V_q^{fin}(\lambda_{i-2j}),$$

If $i = n - 1, n$, then $V_q(i, 1) \cong V_q^{fin}(\lambda_i)$.

(iii) *If $\mathfrak{g}$ is of type C_n, then*

$$V_q(i, 1) \cong V_q^{fin}(\lambda_i),$$

$$V_q(i, 2) \cong \oplus_{j=0}^i V_q^{fin}(2\lambda_j).$$

Proof. The case $m = 1$ was proved in [CP3]. Assume that $\mathfrak{g}$ is of type C_n and that $m = 2$. For C_2, the proposition was proved in [C]. Assume that we know the result for C_{n-1}. Take $J = \{2, \cdots, n\}$. By induction on n, we get

$$\mathbf{U}_{J,q}.v_{i,m} = \bigoplus_{j=1}^{i} V_{J,q}^{fin}(2\lambda_j),$$

(note that we regard $\lambda_j \in P^+$ as an element of P_J^+ by restriction). In other words, there exist vectors $0 \neq w_j \in (\mathbf{U}_{J,q}.w_{i,m})_{2\lambda_j}$ for $1 \leq j \leq i$ with

$$E_{\alpha_r}.w_j = 0 \ \ \forall \, r \in J.$$

Since $2\lambda_i - 2\lambda_j \in \oplus_{i=2}^{n} \mathbf{Z}^+ \alpha_i$, it follows that $E_{\alpha_1}.w_j = 0$ as well. This proves that

$$m_{2\lambda_j}(V(i,m)) = 1, \ \ \forall \ 1 \leq j \leq i,$$

and hence it suffices to prove that the trivial representation occurs in $V_q(i, 2)$. To prove this, let K be the kernel of the map $\phi_2^i : V_q(i, 1) \otimes V_q(i, 1, q^{-2}) \to V_q(i, 2)$ defined in Lemma 2.6. As $\mathbf{U}_q^{fin}$–modules, we have $m_\mu(M) = 1$ if $\mu = 0$ or $\mu = 2\lambda_1$. Let $w_0 \in M$ be such that $E_{\alpha_j}.w_0 = F_{\alpha_j}.w_0 = 0$ for all $j \in I$. Suppose that $w_0 \in K$. Since E_{α_r} and F_{α_0} commute, we must have $F_{\alpha_0}.w_0 = cw_1$ for some $0 \neq c \in \mathbf{C}^\times$. Since $w_1 \notin K$ this means that $c = 0$ and that $\mathbf{C}.w_0$ is the trivial $\mathbf{U}_q$–module. This implies that the modules $V_q(i, 1)$ and $V_q(i, 1, q^2)$ are dual, but it is known, [CP4], that the dual of $V_q(i, 1)$ is the module $V_q(i, 1, q^d)$ where $d \neq 2$ is the Coxeter number of C_n. Hence $w_0 \notin K$ and the multiplicity of the trivial module in $V_q(i, 2)$ is one. This proves the proposition for C_n.

The only remaining case is B_n with $i = n$. But this is proved in the same way as for C_n. We omit the details. $\qquad\blacksquare$

Given an n-tuple of polynomials $\boldsymbol{\pi}_q$, and $J = \{2, \cdots, n\}$, let $\boldsymbol{\pi}_{J,q} = (\pi_2, \cdots, \pi_n)$ and let $V_{J,q}(\boldsymbol{\pi}_{J,q})$ be the irreducible $\mathbf{U}_{J,q}$-module associated to $\boldsymbol{\pi}_{J,q}$. Then, it is easy to see that

$$\mathbf{U}_{J,q}.v_{\boldsymbol{\pi}_q} \cong V_{J,q}(\boldsymbol{\pi}_{J_q}).$$

The next proposition was proved in [CP4], the proof is similar to the one given above for Lemma 2.6.

Proposition 2.5. *The comultiplication Δ of $\mathbf{U}_q$ induces a $\mathbf{U}_{J,q}$-module structure on $\mathbf{U}_{J,q}.v_{\boldsymbol{\pi}_q} \otimes \mathbf{U}_{J,q}.v_{\tilde{\boldsymbol{\pi}}_q}$. Further, the natural map*

$$\mathbf{U}_{J,q}.v_{\boldsymbol{\pi}_q} \otimes \mathbf{U}_{J,q}.v_{\tilde{\boldsymbol{\pi}}_q} \to V_{J,q}(\boldsymbol{\pi}_{J,q}) \otimes V_{J,q}(\tilde{\boldsymbol{\pi}}_{J,q})$$

is an isomorphism of $\mathbf{U}_{J,q}$-modules (the right-hand side is regarded as a $\mathbf{U}_{J,q}$-module by using the comultiplication Δ_J of $\mathbf{U}_{J,q}$) $\qquad\blacksquare$

It is clear from Lemma 2.6 that there exists a $\mathbf{U}_q$-module map $\phi_m^i : V_q(i, 1) \otimes V_q((m-1)\lambda_i, q^{-2}) \to V_q(i, m)$ which maps $v_{i,1} \otimes v_{i,m-1}$ to $v_{i,m}$. For $J = \{2, 3, \cdots, n\}$, let $\phi_{J,m}^i$ denote the analogous map for $\mathbf{U}_{J,q}$. From Proposition 2.5, we see that the restriction of ϕ_m^i to $V_{J,q}(i, 1) \otimes V_{J,q}((m-1)\lambda_i, q^{-2})$ is $\phi_{J,m}^{i-1}$.

In what follows we set $\phi_m = \phi_m^i$ and $\phi_{J,m} = \phi_{J,m}^i$ and we take $J = \{2, 3, \cdots, n\}$. Proposition 2.3 follows from

Proposition 2.6. *Assume that $\mathfrak{g}$ is of type D_n. Let $i \in I$, $i \neq 1, n-1, n$, and $m \geq 0$. For every $\mu \in P(i, m)$, there exist unique (up to scalars) non-zero elements $v_\mu^m \in V_q(i, m)_\mu$ with the following properties.*

$(i)_{m-1}$ *If $\mu_1 \in P_{i,1}$ and $\mu_2 \in P_{i,m-1}$ are such that $\mu_1 + \mu_2 = \mu$, then for some $c_{\mu_1,\mu_2}^{\mu} \in \mathbf{C}^{\times}$,*

$$\phi_m(v_{\mu_1}^1 \otimes v_{\mu_2}^{m-1}) = c_{\mu_1,\mu_2}^{\mu} v_{\mu}^m.$$

$(ii)_m$ *For all $j \in I$,*

$$E_{\alpha_j}.v_{\mu}^m = 0.$$

Further, if $\mu \in P(i,m)$ is such that $\mu + \lambda_2 \in P(i,m)$, then

$$F_{\alpha_0}.v_{\mu}^m = a_{\mu} v_{\mu+\lambda_2}^m,$$

for some $a_{\mu} \in \mathbf{C}^{\times}$.

Analogous statements hold for B_n if $i \neq n$. If $i = n$ or if $\mathfrak{g}$ is of type C_n, then we assume that $m \geq 3$ and in $(i)_{m-1}$ that the element $\mu_1 \in P(i,2)$.

Proof. We begin by remarking that, if elements v_{μ}^m exist with the desired properties, then by Lemma 2.5, they are unique up to scalars. We shall only prove the proposition when $\mathfrak{g}$ is of type D_n, the modifications in the other cases are clear.

Notice that by Lemma 2.5 (since E_{α_r} and F_{α_0} commute for all $r \in I$), if $\mu \in P(i,m)$ is such that $\mu + \lambda_2 \notin P(i,m)$, then

$$F_{\alpha_0}.v_{\mu}^m = 0.$$

Also observe that if $\mu \in P(i,m)$, then $\mu + \lambda_2 \in P(i,m)$ if and only if $\ell(\mu) < m$. We shall use these facts throughout the proof with no further comment.

The statement $(i)_0$ is trivially true. For $(i)_1$ observe that by Lemma 2.7, we have non-zero vectors v_{μ}^1 for $\mu \in P_{i,1}$ such that $E_{\alpha_j}.v_{\mu}^1 = 0$ for all $j \in I$. If i is even, the only element $\mu \in P_{i,1}$ such that $\mu + \lambda_2 \in P_{i,1}$ is $\mu = 0$, and then we have

$$E_{\alpha_r}.v_0^1 = F_{\alpha_r}.v_0^1 = 0 \quad (r \in I).$$

Thus, we have to prove that $F_{\alpha_0}.v_0^1 \neq 0$. But this is clear, since

$$F_{\alpha_0}.v_0^1 = 0 \implies E_{\alpha_0}.v_0^1 = 0,$$

which would imply that v_0^1 generates a proper $\mathbf{U}_q$-submodule of $V_q(i,1)$, contradicting the irreducibility of $V_q(i,1)$. If i is odd, then $\mu + \lambda_2$ is not in $P_{i,1}$ for any $\mu \in P_{i,1}$ and hence the proposition is proved for $m = 1$.

Assume from now on that $(ii)_{m-1}$ and $(i)_{m-1}$ are known for i. We first prove that $(ii)_m$ and $(i)_m$ hold if i is even. For $\mu \in P(i,m)$, let $\mu_1 \in P_{i,1}$ and $\mu_2 \in P_{i,m-1}$ be such that

$$\mu = \mu_1 + \mu_2.$$

Set

$$v_{\mu}^m = \phi_m(v_{\mu_1}^1 \otimes v_{\mu_2}^{m-1}).$$

Then, $v_{\mu}^m \neq 0$ since $(i)_{m-1}$ holds. Clearly,

$$E_{\alpha_j}.v_{\mu}^m = \phi_m(E_{\alpha_j}.(v_{\mu_1}^1 \otimes v_{\mu_2}^{m-1})) = 0.$$

Suppose that $\mu + \lambda_2 \in P(i,m)$, i.e., $\ell(mu) < m$. Then, either $\mu_1 + \lambda_2 \in P_{i,1}$ or $\mu_2 + \lambda_2 \in P_{i,m-1}$. For $j = 1,2$, let $r_j = m - \ell(\mu_j)$. Then, $F_{\alpha_0}^{r_1}.v_{\mu_1}^1 = av_{\mu_1+r_1\lambda_2}^1$ and $F_{\alpha_0}^{r_2}.v_{\mu_2}^{m-1} = bv_{\mu_2+r_2\lambda_2}^{m-1}$ for some non-zero scalars $a, b \in \mathbf{C}(q)$.

Hence,

$$F_{\alpha_0}^{r_1+r_2}.v_{\mu}^m = \phi_m(F_{\alpha_0}^{r_1}v_{\mu_1} \otimes F_{\alpha_0}^{r_2}v_{\mu_2}).$$

Since the right-hand side of the preceding equation is a non-zero scalar multiple of $v_{\mu+(r_1+r_2)\lambda_2}$, it follows that

$$F_{\alpha_0}.v_\mu^m \neq 0.$$

Since $E_{\alpha_r}F_{\alpha_0}.v_\mu^m = 0$ for all $r \in I$, it follows from Lemma 2.5 that $F_{\alpha_0}.v_\mu = a_\mu v_{\mu+\lambda_2}^m$ for some non-zero scalar $a_\mu \in \mathbf{C}(q)$. This shows that $(ii)_m$ holds when i is even. To prove $(i)_m$, let $\mu_1 \in P_{i,1}$ and $\mu_2 \in P(i,m)$ and choose r_1, r_2 so that $\ell(\mu_1+r_1\lambda_2) = 1$ and $\ell(\mu_2 + r_2\lambda_2) = m$. Then,

$$F_{\alpha_0}^{r_1}.v_{\mu_1}^1 = v_{\lambda_2}^1, \quad F_{\alpha_0}^{r_2}.v_{\mu_2}^m = v_{\mu_2+r_2\lambda_2}^m.$$

If $i = 2$, we see that

$$F_{\alpha_0}^{r_1+r_2}.(v_{\mu_1}^1 \otimes v_{\mu_2}^m) = v_{\lambda_2}^1 \otimes v_{m\lambda_2}^m,$$

and hence that

$$\phi_{m+1}(F_{\alpha_0}^{r_1+r_2}.(v_{\mu_1}^1 \otimes v_{\mu_2}^m)) = v_{(m+1)\lambda_2}^{m+1}.$$

Clearly, this implies that $\phi_{m+1}(v_{\mu_1}^1 \otimes v_{\mu_2}^m) \neq 0$, and $(i)_m$ is proved when $i = 2$. In particular, the theorem is proved for $n = 4$.

Assume that we know the proposition for $J = \{2, 3, \cdots, n\}$. Since $m\lambda_i - \mu_2 - r_2\lambda_2 \in Q_J^+$ and $\lambda - \lambda_2 \in Q_J^+$, we see by the induction hypothesis on n that

$$\phi_{m+1}(v_{\lambda_2}^1 \otimes v_{\mu_2+r_2\lambda_2}^m) = \phi_{J,m+1}(v_{\lambda_2}^1 \otimes v_{\mu_2+r_2\lambda_2}^m) \neq 0,$$

i.e., that $\phi_{m+1}(F_{\alpha_0}^{r_1+r_2}.(v_{\mu_1}^1 \otimes v_{\mu_2}^m)) \neq 0$. This implies that $\phi_{m+1}(v_{\mu_1} \otimes v_{\mu_2}) \neq 0$ and proves that $(i)_m$ holds for I.

It remains to prove the result when i is odd; recall that the proposition is known for J. If i is odd, then

$$\mu \in P(i,m) \implies \ell(\mu) = m \implies \mu = m\lambda_i - \eta, \quad (\eta \in Q_J^+).$$

By the induction hypothesis, there exist elements $v_\mu \in \mathbf{U}_{J,q}.v_{i,m}$ satisfying

$$E_{\alpha_j}.v_\mu^m = 0 \quad \text{for all } j \in J.$$

Clearly, $E_{\alpha_1}.v_\mu^m = 0$, and this proves $(ii)_m$ since $\mu + \lambda_2$ is never in $P(i,m)$ if i is odd. To see that $(i)_m$ holds, let $\mu_1 \in P_{i,1}$ and $\mu_2 \in P(i,m)$. Then, $\mu_1 \in \lambda_i - Q_J^+$ and $\mu_2 \in m\lambda_i - Q_J^+$, and hence $v_{\mu_1}^1 \in \mathbf{U}_{J,q}v_{i,1}$ and $v_{\mu_2}^m \in \mathbf{U}_{J,q}.v_{i,m}$. Hence,

$$\phi_{m+1}(v_{\mu_1}^1 \otimes v_{\mu_2}^m) = \phi_{J,m+1}(v_{\mu_1}^1 \otimes v_{\mu_2}^m) \neq 0,$$

thus proving $(i)_m$ when i is odd. The proof of the proposition is now complete. $\qquad\square$

3. The Exceptional Algebras

We summarize here the results that can be proved for the exceptional algebras, using the techniques and results of the previous sections. Again we assume that the nodes are numbered as in [B].

E_6. Here $i \neq 4$.

$$V_q(i, m) = V_q^{fin}(\lambda_i), \quad i = 1, 6,$$
$$V_q(2, m) \cong \bigoplus_{0 \le r \le m} V_q^{fin}(r\lambda_2),$$
$$V_q(3, m) \cong \bigoplus_{r+s=m} V_q^{fin}(r\lambda_3 + s\lambda_6),$$
$$V_q(5, m) \cong \bigoplus_{r+s=m} V_q^{fin}(r\lambda_5 + s\lambda_1).$$

E_7. Here $i = 1, 2, 6, 7$.

$$V_q(1, m) \cong \bigoplus_{0 \le r \le m} V_q^{fin}(r\lambda_1),$$
$$V_q(7, m) \cong V_q^{fin}(\lambda_7),$$
$$V_q(2, m) \cong \bigoplus_{r+s=m} V_q^{fin}(r\lambda_2 + s\lambda_7),$$
$$V_q(6, m) \cong \bigoplus_{0 \le r+s \le m} V_q^{fin}(r\lambda_6 + s\lambda_1).$$

E_8. Here $i = 1, 8$.

$$V_q(1, m) \cong \bigoplus_{0 \le r \le m} V_q^{fin}(r\lambda_8),$$
$$V_q(8, m) \cong \bigoplus_{0 \le r+s \le m} V_q^{fin}(r\lambda_1 + s\lambda_8).$$

w
F_4.

$$V_q(1, m) \cong \bigoplus_{k=0}^{m} V_q^{fin}(s\lambda_1),$$
$$V_q(4, m) \cong \bigoplus_{j=0}^{k} \oplus_{k=0}^{m/2} V_q^{fin}(j\lambda_1 + (m - 2k)\lambda_4).$$

G_2.

$$V_q(1, m) \cong \bigoplus_{k=0}^{m} V_q^{fin}(k\lambda_1).$$

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